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Semiclassical Corrections to the Oscillation Frequencies of a Trapped Bose-Einstein Condensate

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Abstract

The oscillation frequencies of collective excitations of a trapped Bose-Einstein condensate, when calculated in the mean-field approximation and in the Thomas-Fermi limit, are independent of the scattering length a . We calculate the leading corrections to the frequencies from quantum fluctuations around the mean field. The semiclassical correction is proportional to $N^{1/5}a^{6/5}$, where N is the number of atoms in the condensate. The correction is positive semidefinite and is zero for surface modes whose eigenfunctions have a vanishing laplacian. The shift in the frequency of the lowest quadrupole mode for an axially symmetric trap is large enough that it should be measurable in future experiments.

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The achievement of Bose-Einstein condensation in trapped atomic vapors [1] has opened up new possibilities for the experimental study of nonhomogeneous interacting Bose gases. This system can be conveniently described by a quantum field theory, but the quantum field equations are extremely difficult to solve in general. Fortunately, many of the basic properties of the condensates in existing experiments are described accurately by the mean-field approximation, which reduces the problem to solving the Gross-Pitaevskii equation, a classical equation for the mean field. These properties include the density profile of the ground state and collective oscillations of the condensate. Fluctuations of the quantum field around the mean field provide corrections to mean-field predictions that scale as the square root of the peak number density of the atoms. These corrections should be observable through precise measurements of the properties of condensates containing a large number of atoms.

Among the most beautiful experiments on Bose-Einstein condensates are the excitation and study of collective oscillations of the condensates. Among the properties of the collective modes that have been measured are the frequencies of the lowest normal modes at extremely low temperatures [2,3,5], the temperature dependence of the frequencies and their damping rates, [5,4], and the speed of sound [6]. The frequencies of the normal modes can be measured very accurately. In one experiment, they have been measured to better than a fraction of a percent [5]. These frequencies are therefore an ideal place to look for the effects of quantum field fluctuations.

Most previous studies of the collective excitations of a trapped condensate at zero temperature have been carried out using the mean-field approximation or the Bogoliubov approximation, which is equivalent at $T = 0$. The collective excitations have been studied by variational methods [7], by numerical methods [8], and by analytic methods [9,10]. In the Thomas-Fermi limit, the oscillation frequencies for a condensate in a harmonic potential can be obtained analytically and are independent of the interaction strength of the atoms and the number of atoms in the trap [9]. Quantum fluctuations around the mean field give fractional corrections to the frequencies that are proportional to $\sqrt{\rho a^3}$, where ρ is the peak

number density and a is the S-wave scattering length of the atoms. The purpose of this paper is to calculate these corrections.

A Bose-Einstein condensate containing a large number N of identical atoms in a trapping potential $V(\mathbf{r})$ can be described by a quantum field theory with a single complex-valued field $\psi(\mathbf{r}, t)$ that satisfies

$$i\hbar\dot{\psi} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V - \mu \right] \psi + \frac{\hbar^2 g}{4m}(\psi^\dagger\psi)\psi, \quad (1)$$

where $g = 16\pi a$. The chemical potential μ in (1) is to be adjusted so that the expectation value of $\int d^3r \psi^\dagger\psi$, the total number of atoms, is equal to N . A collective oscillation of the Bose-Einstein condensate is described by a time-dependent state in the quantum field theory. Letting $\langle \dots \rangle$ denote the expectation value in that state, we can write the quantum field as $\psi = \phi + \tilde{\psi}$, where $\phi = \langle \psi \rangle$ is the mean field and $\tilde{\psi}$ is the quantum fluctuation field that satisfies $\langle \tilde{\psi} \rangle = 0$. The number density ρ and the current density \mathbf{j} in that state are

$$\rho = |\phi|^2 + \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle, \quad (2)$$

$$\mathbf{j} = -i\frac{\hbar}{2m}(\phi^*\nabla\phi - \phi\nabla\phi^*) - i\frac{\hbar}{2m}\langle \tilde{\psi}^\dagger\nabla\tilde{\psi} - \nabla\tilde{\psi}^\dagger\tilde{\psi} \rangle. \quad (3)$$

The continuity equation $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$ follows from the U(1) symmetry of the field theory. Taking the expectation value of (1), we obtain a partial differential equation for ϕ that involves the matrix elements of $\tilde{\psi}$:

$$i\hbar\dot{\phi} + \left[-\frac{\hbar^2}{2m}\nabla^2 + V - \mu \right] \phi + \frac{\hbar^2 g}{4m} \left[|\phi|^2\phi + 2\langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \phi + \langle \tilde{\psi} \tilde{\psi} \rangle \phi^* + \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \right]. \quad (4)$$

In the mean-field approximation, the matrix elements are neglected and (4) reduces to the Gross-Pitaevskii equation. The effects of quantum field fluctuations are contained in the matrix elements, which can be expressed as functionals of the mean field ϕ . They are dominated by modes of $\tilde{\psi}$ with wavelengths comparable to the local coherence length ξ , which is the scale of the wavelength above which the excitations of the condensate are modified by collective effects. In the Thomas-Fermi limit, ξ is much smaller than the length scale for significant variations in ϕ . The matrix elements can therefore be expanded in derivatives

of ϕ by generalizing the methods developed in Refs. [11,12] to the case of a time-dependent state. Including terms through first order in $g^{3/2}|\phi|$ and through first order in derivatives of ϕ , we obtain [13]

$$\langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \approx \frac{1}{24\pi^2} g^{3/2} |\phi|^3, \quad (5)$$

$$\langle \tilde{\psi} \tilde{\psi} \rangle \approx \frac{1}{8\pi^2} g^{3/2} |\phi| \phi^2 - \frac{im}{8\pi^2 \hbar} g^{1/2} |\phi|^{-3} \phi^2 (\phi^* \dot{\phi} + \dot{\phi}^* \phi), \quad (6)$$

$$\langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \approx 0, \quad (7)$$

$$\langle \tilde{\psi}^\dagger \nabla \tilde{\psi} - \nabla \tilde{\psi}^\dagger \tilde{\psi} \rangle \approx \frac{1}{24\pi^2} g^{3/2} |\phi| (\phi^* \nabla \phi - \phi \nabla \phi^*). \quad (8)$$

To verify the continuity equation $\dot{\rho} + \nabla \cdot \mathbf{j} = 0$ to first order in $g^{3/2}|\phi|$ and to second order in gradients of ϕ , we must include terms in $\langle \tilde{\psi}^\dagger \tilde{\psi} \rangle$ and $\langle \tilde{\psi} \tilde{\psi} \rangle$ that are second order in gradients of ϕ . These terms are rather complicated [12], but fortunately they are not needed to derive the equation for the time evolution of ρ in the Thomas-Fermi limit. By differentiating the continuity equation with respect to time, using (4) to eliminate $\dot{\phi}$ from the time derivative of \mathbf{j} , and using (2) to eliminate ϕ in favor of ρ , we can obtain a closed equation for ρ . Keeping only terms through first order in $(g^3 \rho)^{1/2}$ and through second order in gradients of ρ , this equation is

$$\ddot{\rho} = \frac{\hbar^2 g}{4m^2} \nabla \cdot \left\{ \rho \nabla \left[\rho - \frac{4m}{\hbar^2 g} (\mu - V) + \frac{1}{6\pi^2} g^{3/2} \rho^{3/2} \right] \right\}. \quad (9)$$

This equation gives the time-dependence of the density in the Thomas-Fermi limit and to first order in the semiclassical expansion. The ground state number density ρ_0 in this approximation satisfies

$$\left[1 + \frac{1}{6\pi^2} (g^3 \rho_0)^{1/2} \right] \rho_0 = \frac{4m}{\hbar^2 g} (\mu - V). \quad (10)$$

To deduce the equations for small-amplitude oscillations of the condensate, we substitute $\rho = \rho_0 + \tilde{\rho}$ into (9) and linearize in $\tilde{\rho}$:

$$\ddot{\tilde{\rho}} = \frac{\hbar^2 g}{4m^2} \nabla \cdot \left\{ \rho_0 \nabla \left[\tilde{\rho} + \frac{1}{4\pi^2} g^{3/2} \rho_0^{1/2} \tilde{\rho} \right] \right\}. \quad (11)$$

In the mean-field approximation, the $(g^3 \rho_0)^{1/2}$ corrections in (10) and (11) are neglected. The solution for the ground state number density $\bar{\rho}_0$ vanishes at points \mathbf{r} where $V(\mathbf{r}) > \mu$

and is given by $\bar{\rho}_0 = 4m(\mu - V)/\hbar^2 g$ at points where $V(\mathbf{r}) < \mu$. Assuming that $\tilde{\rho}$ has the time dependence $\exp(i\omega t)$, we obtain the eigenvalue equation $H_0\tilde{\rho} = \bar{\omega}^2\tilde{\rho}$, where

$$H_0 = -\frac{\hbar^2 g}{4m^2} \nabla \cdot (\bar{\rho}_0 \nabla). \quad (12)$$

The “hamiltonian” H_0 , whose eigenvalues give the mean-field approximations to the oscillation frequencies in the Thomas-Fermi limit, was first derived by Stringari [9].

To obtain the semiclassical correction to the frequencies, we expand (11) to first order in $(g^3\bar{\rho}_0)^{1/2}$. The solution of (10) to first order in $(g^3\bar{\rho}_0)^{1/2}$ is $\rho_0 = \bar{\rho}_0[1 - (g^3\bar{\rho}_0)^{1/2}/6\pi^2]$. Inserting this into (11) and assuming that $\tilde{\rho}$ has the time-dependence $\exp(i\omega t)$, we obtain an eigenvalue equation for ω . Absorbing a factor of $1 + g^{3/2}\bar{\rho}_0^{1/2}/(8\pi^2)$ into $\tilde{\rho}$, our equation becomes the eigenvalue equation for a hermitian operator: $(H_0 + H_1)\tilde{\rho} = \omega^2\tilde{\rho}$, where

$$H_1 = -\frac{\hbar^2 g^{5/2}}{96\pi^2 m^2} (\bar{\rho}_0^{3/2} \nabla^2 + \nabla^2 \bar{\rho}_0^{3/2}). \quad (13)$$

The eigenvalues of $H_0 + H_1$ give the oscillation frequencies in the Thomas-Fermi limit through first order in the semiclassical expansion. The semiclassical corrections to the frequencies can be obtained by using the standard results of time-independent perturbation theory. If $\bar{\omega}_n^2$ is a nondegenerate eigenvalue of H_0 with eigenfunction $\tilde{\rho}_n$, then the semiclassical correction $\delta\omega_n^2 = \omega_n^2 - \bar{\omega}_n^2$ is

$$\delta\omega_n^2 = -\frac{\hbar^2 g^{5/2}}{48\pi^2 m^2} \frac{\text{Re} \int d^3r \bar{\rho}_0^{3/2} \tilde{\rho}_n^* \nabla^2 \tilde{\rho}_n}{\int d^3r |\tilde{\rho}_n|^2}. \quad (14)$$

If the eigenvalue $\bar{\omega}_n^2$ is degenerate, then the expression (14) holds only in the basis consisting of states that diagonalize the restriction of H_1 to the subspace of states with eigenvalue $\bar{\omega}_n^2$. Note that the correction (14) is positive or zero. This follows from the fact that H_1 is proportional to the anticommutator of the positive operator $\bar{\rho}_0^{3/2}$ and the positive-semidefinite operator $-\nabla^2$. The correction (14) is zero only if the laplacian of the eigenfunction $\tilde{\rho}_n$ vanishes.

We first consider a condensate trapped in the spherically symmetric parabolic potential $V(r) = m\omega_0^2 r^2/2$. The ground state number density $\bar{\rho}_0$ in the mean-field approxi-

mation and in the Thomas-Fermi limit vanishes outside a sphere of radius R given by $N = m^2 \omega_0^2 R^5 / (15 \hbar^2 a)$. Inside that sphere, it has the form

$$\bar{\rho}_0(r) = \frac{2m^2 \omega_0^2}{\hbar^2 g} (R^2 - r^2), \quad r < R. \quad (15)$$

The normal modes for oscillation of the condensate around the ground state can be labelled by a radial quantum number n and angular momentum quantum numbers ℓ and m . The mean-field frequencies $\bar{\omega}_{n\ell}$ depend on the quantum numbers n and ℓ , and were first determined by Stringari [9]:

$$\bar{\omega}_{n\ell}^2 = [2n^2 + 3n + (2n + 1)\ell] \omega_0^2. \quad (16)$$

The $n = 0$ modes are surface modes and their eigenfunctions are $r^\ell Y_{\ell m}(\theta, \phi)$. Since they have a vanishing laplacian, the semiclassical correction to the frequencies of these surface modes vanishes: $\delta \omega_{0\ell}^2 = 0$. The eigenfunctions for $n = 1$ are $(R^2 - \frac{2\ell+5}{2\ell+3} r^2) r^\ell Y_{\ell m}(\theta, \phi)$ for $r < R$. Inserting this into (14), we find that the shifts in the frequencies are

$$\delta \omega_{1\ell}^2 = \frac{3(2\ell + 3)(\frac{1}{2})_{\ell+4}}{\sqrt{2}(\ell + 4)!} \left(\frac{m \omega_0 a R}{\hbar} \right) \omega_0^2, \quad (17)$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol. The fractional shift in the frequency is proportional to $m \omega_0 a R / \hbar$, which scales like $N^{1/5} a^{6/5} \omega_0^{3/5}$. It therefore increases very slowly with the number of atoms in the condensate.

We next consider a condensate trapped in the axially symmetric parabolic potential $V(\rho, z) = m(\omega_\perp^2 \rho^2 + \omega_\parallel^2 z^2)/2$. The ground state number density in the mean-field approximation and in the Thomas-Fermi limit takes the form

$$\bar{\rho}_0(\mathbf{r}) = \frac{2m^2 \omega_\perp^2}{\hbar^2 g} (R^2 - \rho^2 - \lambda^2 z^2), \quad \rho^2 + \lambda^2 z^2 < R^2, \quad (18)$$

where $\lambda = \omega_\parallel / \omega_\perp$ and R is given by $N = m^2 \omega_\perp^2 R^5 / (15 \hbar^2 a \lambda)$. The eigenfunctions have the form $\rho^{|m|} e^{im\phi} P(z, \rho^2)$, where m is the azimuthal angular momentum quantum number and P is a polynomial in z and ρ^2 . The number of independent solutions for which P is a k 'th order polynomial in z is $\text{int}(1 + k/2)$. For $k = 0$ and $k = 1$, the frequencies

in the mean-field approximation are $\bar{\omega}_{0m}^2 = |m|\omega_\perp^2$ and $\bar{\omega}_{1m}^2 = (|m| + \lambda^2)\omega_\perp^2$, respectively. The corresponding eigenfunctions are $\rho^{|m|}e^{im\phi}$ and $z\rho^{|m|}e^{im\phi}$. Since these eigenfunctions have vanishing laplacian, the semiclassical corrections to the frequencies vanish: $\delta\omega_{0m}^2 = \delta\omega_{1m}^2 = 0$. For each value of m , there are two independent eigenfunctions with $k = 2$. Their frequencies in the mean-field approximation are

$$\bar{\omega}_{2m\pm}^2 = \left(2(|m| + 1) + \frac{3}{2}\lambda^2 \pm \sqrt{(|m| + 2)^2 - (|m| + 4)\lambda^2 + \frac{9}{4}\lambda^4}\right)\omega_\perp^2. \quad (19)$$

These were first derived by Stringari for $m = 0$ [9] and by Ohberg et al. and by Fliesser et al. for general m [10]. The eigenfunctions have the form $(R^2 + \alpha_{m\pm}\rho^2 + \beta_{m\pm}\lambda^2 z^2)\rho^{|m|}e^{im\phi}$, where $\alpha_{m\pm}$ and $\beta_{m\pm}$ are complicated functions of $|m|$ and λ . Using (14), we find that the semiclassical corrections to the frequencies are

$$\begin{aligned} \delta\omega_{2m\pm}^2 &= \frac{3(\frac{1}{2})_{|m|+4}f_\pm(|m|, \lambda)}{\sqrt{2}(|m| + 4)!} \left(\frac{m\omega_\perp a R}{\hbar}\right)\omega_\perp^2, \\ f_\pm(|m|, \lambda) &= |m| + 1 + \frac{1}{2}\lambda^2 \pm \frac{(|m| + 1)(|m| + 2) - \frac{1}{2}\lambda^2 + \frac{3}{4}\lambda^4}{\sqrt{(|m| + 2)^2 - (|m| + 4)\lambda^2 + \frac{9}{4}\lambda^4}}. \end{aligned} \quad (20)$$

If we set $\lambda = 1$, this reduces to (17) with $\ell \rightarrow |m|$.

The most accurate measurement to date of an oscillation frequency for a Bose-Einstein condensate was carried out by Stamper-Kurn et al. [5]. They trapped $N = 1.5 \times 10^7$ atoms of ^{23}Na , whose scattering length is $a = 2.75$ nm, in an axially symmetric potential. The axial trapping frequency was measured to be $\omega_\parallel/2\pi = 16.93(2)$ Hz and the transverse trapping frequency was estimated to be $\omega_\perp/2\pi = 230$ Hz, so that $\lambda \approx 0.074$. They measured the ratio $\omega_{20-}/\omega_\parallel$ of the lowest quadrupole oscillation frequency to the axial trap frequency to be $1.569(4)$. The fractional uncertainty in the measurement is 0.25%. The mean-field prediction from (19) is $\bar{\omega}_{20-}/\omega_\parallel = 1.5806$, which is larger by about 3 standard deviations. Including the semiclassical correction (20), we obtain the prediction 1.5813. The fractional increase is 0.04%, which is about $\frac{1}{6}$ of a standard deviation. Thus there is a several standard deviation discrepancy between the prediction and the measurement.

In order to assess the significance of the discrepancy, we must consider other sources of corrections to the frequency, including edge effects, nonlinear effects, and nonzero temper-

ature effects. The correction from edge effects arises from the breakdown of the Thomas-Fermi approximation near the edge of the condensate. The fractional correction scales like $(\hbar/m\omega R^2)^2$ [14], which is approximately 4% for the experiment in Ref. [5], so these corrections are probably not negligible. This source of theoretical error can be eliminated by avoiding the Thomas-Fermi limit, and instead calculating the frequencies numerically by solving the mean-field equations. The frequency shift from nonlinear effects has been studied by Dalfovo et al. [15]. The fractional shift for the lowest quadrupole mode was found to be $+0.09A^2$, where A is the relative magnitude of the oscillation in the length of the condensate. Taking the value $A \approx 0.1$ from Ref. [5], we find that the fractional frequency shift is 0.9%. This source of error can be eliminated experimentally by decreasing the amplitude of the excitation of the condensate. It was suggested in Ref. [5] that the discrepancy in the frequency can be explained by nonzero temperature corrections, which have the correct sign. While there have been studies of the temperature dependence of the frequencies [16], there don't seem to be any quantitative treatments for the low temperatures appropriate to the experiment of Ref. [5].

We have calculated the semiclassical corrections to the frequencies for small oscillations of a Bose-Einstein condensate in the Thomas-Fermi limit. The frequency shifts are less than an order of magnitude smaller than the uncertainties in some previous frequency measurements. These shifts should therefore be measurable by more accurate experiments. Accurate theoretical predictions require going beyond the Thomas-Fermi limit in calculating the semiclassical frequency shifts. Separating the semiclassical correction from that due to nonzero temperature may require the development of new techniques for measuring the temperatures of extremely cold condensates. A measurement of the semiclassical frequency shift in agreement with theoretical predictions would mark a new threshold in our quantitative understanding of Bose-Einstein condensed gases.

While this work was being prepared for publication, a preprint by Pitaevskii and Stringari [17] appeared in which the authors also calculate the semiclassical corrections to the oscillation frequencies of a trapped Bose-Einstein condensate. Their results are in perfect

agreement with ours.

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REFERENCES

- [1] M.H. Anderson et al., Science **269**, 198 (1995); C.C. Bradley et al., Phys. Rev. Lett. **75**, 1687 (1995); K.B. Davis et al., Phys. Rev. Lett. **75**, 1969 (1995).
- [2] D.S. Jin et al., Phys. Rev. Lett. **77**, 420 (1996).
- [3] M.O. Mewes et al., Phys. Rev. Lett. **77**, 988 (1996).
- [4] D.S. Jin et al., Phys. Rev. Lett. **77**, 764 (1996).
- [5] D.M. Stamper-Kurn et al., cond-mat/9801262.
- [6] M.R. Andrews et al., Phys. Rev. Lett. **79**, 553 (1997).
- [7] A. L. Fetter, Phys. Rev. A **53**, 4245 (1996); K.G. Singh and D.S. Rokhsar, Phys. Rev. Lett. **77**, 1667 (1996); V.M. Pérez-García et. al., Phys. Rev. Lett. **77**, 5320 (1996); Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **55**, R18 (1997).
- [8] P.A. Ruprecht et al., Phys. Rev. A **54**, 4178 (1996); M. Edwards et al., Phys. Rev. Lett. **77**, 1671 (1996); B.D. Esry, Phys. Rev. A **55**, 1147 (1997); L. You, W. Hoston, and M. Lewenstein, Phys. Rev. A **55**, R1581 (1997); D.A.W. Hutchinson and E. Zaremba, Phys. Rev. A **57**, 1280 (1998).
- [9] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
- [10] P. Öhberg et al., Phys. Rev. A **56**, R3346 (1997); M. Fliesser et al., Phys. Rev. A **56**, R2533 (1997).
- [11] E. Braaten and A. Nieto, Phys. Rev. B **56**, 14745 (1997).
- [12] J.O. Andersen and E. Braaten, cond-mat/9806375.
- [13] J.O. Andersen and E. Braaten (in preparation).
- [14] A.L. Fetter and D.L. Feder, Phys. Rev. A **58**, 3185 (1998).
- [15] F. Dalfovo, C. Minniti, and L.P. Pitaevskii, Phys. Rev. A **56**, 4855 (1997).

- [16] For a review, see F. Dalfovo, S. Giorgino, L.P. Pitaevskii and S. Stringari, cond-mat/9806038.
- [17] L. Pitaevskii and S. Stringari, cond-mat/9807396.